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The Principle of Linearized Stability for a Class of Degenerate Diffusion Equations

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$. Consider the initial-boundary value problem

$$\text{(I)} \begin{cases} (\beta(v))_t = \Delta v + f(v) & \text{in} \quad \Omega \times \mathbb{R}^+, \\ v = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}^+ \\ v(\cdot, 0) = v_0 & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here v_0 is a given bounded nonnegative function, the functions $\beta(s)$ and $f(s)$, defined for $s \geqslant 0$, are smooth for $s > 0$, $f(0) \geqslant 0$, $\beta(0) = 0$, $\beta'(0) = +\infty$, $\beta'(s) > 0$ for $s > 0$ and $f \circ \beta^{-1}(s)$ is Lipschitz continuous for $s \geqslant 0$ (the precise hypotheses on the data can be found in Section 2). Since $\beta'(0) = \infty$, the diffusion is degenerate near points where $v = 0$.

In this paper we study the stability of steady-state solutions of Problem I by linearizing (1.1) near an equilibrium $\bar{v}$. So let $\bar{v} \in C^2(\bar{\Omega})$ satisfy

$$\text{(II)} \begin{cases} \Delta \bar{v} + f(\bar{v}) = 0 & \text{in} \quad \Omega, \\ \bar{v} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We assume throughout that

$$\bar{v} \geqslant 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

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Actually we shall primarily restrict ourselves to the case when

$$\bar{v} > 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \frac{\partial \bar{v}}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\nu(x)$ denotes the outward normal at $x \in \partial\Omega$. In the final section, Section 5, we shall make some remarks about the case when $\bar{v}$ merely satisfies (1.2) instead of (1.3).

To introduce the concept of linearized stability, we linearize (1.1), formally, around $\bar{v}$ to arrive at

$$\beta'(\bar{v})\, z_t = \Delta z + f'(\bar{v})\, z.$$

The behaviour of solutions $z(x, t)$ depends on the spectrum of the corresponding eigenvalue problem

$$\text{(III)} \begin{cases} -\Delta w = f'(\bar{v})\, w + \lambda \beta'(\bar{v})\, w & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It turns out that, due to condition (1.3), Problem III possesses a discrete spectrum of eigenvalues $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$. If $\lambda_1 > 0$ then $z(\cdot, \;)$ decays like $e^{-\lambda_1 t}$ as $t \to \infty$. We show that this behaviour is preserved in the original nonlinear problem in the following sense: if the initial function v_0 of Problem I is nonngative and if

$$\|v_0 - \bar{v}\|_{L^\infty(\Omega)} \text{ is sufficiently small} \tag{1.4}$$

then the solution $v(x, t; v_0)$ of Problem I satisfies

$$|v(x, t; v_0) - \bar{v}(x)| \leqslant C e^{-\lambda_1 t} e(x), \qquad x \in \bar{\Omega}, \quad t \geqslant T, \tag{1.5}$$

for some positive constants C and T. Here $e \in C^2(\bar{\Omega})$ is defined by

$$-\Delta e = 1 \quad \text{in} \quad \Omega, \qquad e = 0 \quad \text{on} \quad \partial\Omega. \tag{1.6}$$

If on the contrary $\lambda_1 < 0$, then $\bar{v}$ is unstable and repels some neighbouring orbits. For the precise statements see the Theorems 4.1 and 4.8.

The proof of (1.5), given in Section 4, is fairly long and technical. This is partly due to the choice of the L^∞ topology in (1.4). The use of a stronger topology would considerably simplify the proofs. However, this would be at the expense of reducing the practical value of the conclusions. In particular the choice of the L^∞ norm is in agreement with a result of Di Benedetto [9], that the orbit $\{v(\cdot, t): t \geqslant \tau > 0\}$ is relatively compact in $C(\bar{\Omega})$ if it is bounded in $L^\infty(\Omega)$.

As we indicated already, another source of complications is the singular eigenvalue problem III. In Section 3 we study the more general eigenvalue problem

$$-\Delta w = c(x)\, w + \lambda b(x) w \qquad \text{in} \quad \Omega,$$
$$w = 0 \qquad \text{on} \quad \partial\Omega,$$

where the coefficients b and c are allowed to blow up near $\partial\Omega$ at a controlled rate. It will turn out that

$$\{\operatorname{dist}(x, \partial\Omega)\}^{-2}$$

is the critical growth condition for b and c. When the blow-up rate of b and c is slower than the critical one, we prove the existence of an unbounded sequence of discrete eigenvalues. We also derive some results when b and c do not satisfy a growth condition near $\partial\Omega$. Section 3 is completely self-contained.

In Section 5 we shall give an application of our linearization result to the equation

$$u_t = \Delta(u^m) + u^p, \qquad 1 \leqslant p < m,$$

and we shall comment on situations where the idea of linearization fails.

There is an extensive literature on the study of stability via linearization for semilinear and quasilinear nondegenerate equations. See, for example, Sattinger [22] for the linearization of the general Navier–Stokes equations near a steady-state solution, Kirchgässner and Kielhöfer [11] for the Bénard and Taylor problem, Sattinger [21] and P. L. Lions [15, 16] for semilinear parabolic equations, Potier-Ferry [18] for quasilinear parabolic equations, Klainerman [12] for quasilinear wave equations and Henry [10] for a general discussion and references to the literature.

For a general sltudy of elliptic problems of type II we refer to the survey article of P. L. Lions [17], which contains an extensive list of references.

2. Preliminaries

Throughout this paper we shall use the following hypotheses about the data Ω, β, f and v_0:

(H1) Ω is a bounded, open and connected subset of $\mathbb{R}^N$ ($N \geqslant 1$) whose boundary is of class C^3.

(H2) $\beta \in C^3(\mathbb{R}^+) \cap C(\overline{\mathbb{R}^+})$, $\beta(0) = 0$, $\beta'(s) > 0$ for $s > 0$, $\lim_{s \searrow 0} \beta'(s) = +\infty$ and

$$-\beta'(s) \leqslant s\beta''(s) \leqslant 0 \qquad \text{for} \quad 0 < s \leqslant s_0$$

for some positive constant s_0.

(H3) $f \in C^2(\mathbb{R}^+) \cap C^\alpha(\overline{\mathbb{R}^+})$ for some $\alpha \in (0, 1)$, $f(0) \geqslant 0$, if $f(0) > 0$ then $f \in C^2(\overline{\mathbb{R}^+})$, $f \circ \beta^{-1}$ is uniformly Lipschitz continuous on $\overline{\mathbb{R}^+}$,

$$-K\beta'(s) \leqslant sf''(s) \leqslant K\beta'(s) \qquad \text{for} \quad 0 < s \leqslant s_0$$

for some $K > 0$ and

$$sf'(s) \in C^\alpha(\overline{\mathbb{R}^+})$$

(here we define $sf'(s)$ at $s = 0$ by $\lim_{s \searrow 0} sf'(s) = 0$).

(H4) $v_0 \in L^\infty$ and $v_0 \geqslant 0$ a.e. in Ω.We shall refer to these hypotheses collectively as hypothesis (H).

Consider the problem

$$(\mathrm{IV}) \begin{cases} \beta(v)_t = \Delta v + f(v) & \text{in} \quad \Omega \times \mathbb{R}^+, \\ v = \chi & \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ v(\cdot, 0) = v_0 & \text{in} \quad \Omega, \end{cases}$$

where $\chi \in L^\infty(\partial\Omega \times \mathbb{R}^+)$ is a nonnegative function.

Let $T > 0$, $Q_T = \Omega \times (0, T]$ and $Q = \Omega \times \mathbb{R}^+$.

Definition 2.1. A function v, defined and nonnegative a.e. in $\bar{Q}$, is called a solution of Problem IV if

(i) $\beta(v) \in C([0\ T]: L^1(\Omega)) \cap L^\infty(Q_T)$ for any $T > 0$;

(ii)

$$\int_\Omega \beta(v(T))\, \eta(T) = \int_\Omega \beta(v_0)\, \eta(0) + \iint_{Q_T} \{\beta(v)\, \eta_t + v\Delta\eta + f(v)\, \eta\} - \iint_{\partial\Omega \times (0,T)} \chi \frac{\partial \eta}{\partial \nu} \tag{2.1}$$

for any $T > 0$ and $\eta \in C^2(\bar{Q})$ such that $\eta \geqslant 0$ in Q and $\eta = 0$ on $\partial\Omega \times \mathbb{R}^+$.

A subsolution (supersolution) of Problem IV is defined by (i) and (ii) with equality in (2.1) replaced by $\leqslant$ ($\geqslant$).

We shall need the following basic results about Problem I and Problem IV.

Proposition 2.2. *Let Ω, f, β and v_0 satisfy hypothesis* (H). *Then Problem* I *possesses a unique solution.*

Proposition 2.3 (Comparison Principle). *Let $\bar{v}$ be a supersolution of Problem* IV *with data $\bar{v}_0$, $\bar{\chi}$ and let $\underline{v}$ be a subsolution of Problem* IV *with* *data* $\underline{v}_0, \chi$ *and let* Ω, β, f, $\bar{v}_0$ *and* $\underline{v}_0$ *satisfy hypothesis* (H). *If* $\underline{\chi} \leqslant \bar{\chi}$ *and* $\underline{v}_0 \leqslant \bar{v}_0$, *then*

$$\underline{v}(t) \leqslant \bar{v}(t) \qquad \text{a.e. in } \Omega \text{ for all } t \geqslant 0.$$

For the proof of these propositions we refer to [2].

Finally, we mention a property of the function β.

LEMMA 2.4. *Let* β *satisfy* (H2). *Then*

$$s_1 \beta'(s_1) \leqslant s_2 \beta'(s_2) \qquad \text{for} \quad 0 < s_1 \leqslant s_2 \leqslant s_0.$$

Proof. Differentiating $s\beta'(s)$ yields, using (H2),

$$(s\beta'(s))' = \beta'(s) + s\beta''(s) \geqslant 0, \qquad s \in (0, s_0],$$

and Lemma 2.4 follows at once.

3. A SINGULAR EIGENVALUE PROBLEM

In this section we study the singular eigenvalue problem

$$-\Delta w = cw + \lambda bw, \qquad w \in H_0^1(\Omega), \tag{3.1}$$

where the functions b and c satisfy

$$\begin{aligned} &b, c \in L^\infty_{\text{loc}}(\Omega), \qquad b(x) \geqslant b_0 > 0, \\ &|c(x)| \leqslant K_1 b(x) \qquad \text{for} \quad \text{a.e. } x \in \Omega, \end{aligned} \tag{3.2}$$

and

$$b(x) \leqslant K_2 \{d_\Omega(x)\}^{-2} \tag{3.3}$$

for some positive constants b_0, K_1 and K_2. Here

$$d_\Omega(x) = \text{distance}(x, \partial\Omega), \qquad x \in \Omega.$$

The following lemma, applied with $q = 2$, shows that, in view of (3.3), Eq. (3.1) is well defined in $H_0^1(\Omega)$.

LEMMA 3.1. *Let* $w \in W_0^{1,q}(\Omega)$ *for some* $1 < q < \infty$. *Then there exists a constant* $C > 0$ *such that*

$$\left\| \frac{w}{d_\Omega} \right\|_{L^q(\Omega)} \leqslant C \, \|w\|_{W_0^{1,q}(\Omega)}.$$

The proof of Lemma 3.1 can be found in Kufner [13, p. 69, Theorem 8.4].

First we shall derive some results about the existence of infinitely many eigenvalues of Problem (3.1), i.e., the values of λ for which (3.1) has a nontrivial solution. In Theorem 3.4 we remove the growth condition (3.3) and give a sufficient condition for the existence of at least one eigenvalue.

THEOREM 3.2. *Let b and c satisfy* (3.2) *and let*

$$d_\Omega^2(x)\, b(x) \to 0 \qquad \textit{uniformly as } d_\Omega(x) \to 0. \tag{3.4}$$

Then Problem (3.1) *has countably many eigenvalues $\lambda_1 < \lambda_2 \leqslant \lambda_3, \ldots,\ \lambda_i \to \infty$ as $i \to \infty$. The corresponding eigenfunctions $w_i \in H_0^1(\Omega)$ satisfy*

$$\int_\Omega b w_i w_j = 0 \qquad \textit{if} \quad \lambda_i \neq \lambda_j,$$

and $w_i \in W_{\mathrm{loc}}^{2,p}(\Omega)$ for $1 \leqslant p < \infty$. If $w_1(x_0) > 0$ for some $x_0 \in \Omega$, then

$$w_1(x) > 0 \qquad \textit{for} \quad x \in \Omega. \tag{3.5}$$

For the proof of Theorem 3.2 we introduce the weighted space $L^2(\Omega; b)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|w\|_{L^2(\Omega;b)} = \left\{ \int_\Omega b w^2 \right\}^{1/2}.$$

By Lemma 3.1, $H_0^1(\Omega) \subset L^2(\Omega; b)$ if b satisfies (3.2) and (3.3). The next lemma shows that if b satisfies (3.4), the imbedding is compact.

LEMMA 3.3. *Let b satisfy* (3.2) *and* (3.4). *Then $H_0^1(\Omega)$ is compactly imbedded into $L^2(\Omega; b)$.*

Using this compactness result, the proof of Theorem 3.2 is straightforward and follows the same lines as in the case $b \equiv 1$. To derive (3.5), the strong maximum principle in $W_{\mathrm{loc}}^{2,p}(\Omega)$ $(p > N)$ [7] can be used. So it remains to prove Lemma 3.3.

The basic idea of this proof, due to R. M. Kaufman, is also used in Lemma 1 of Lewis [14].

Proof of Lemma 3.3. Let

$$\Omega_\varepsilon = \{x \in \Omega \colon d_\Omega(x) > \varepsilon\}, \qquad \varepsilon > 0.$$

We define the map $I_\varepsilon \colon H_0^1(\Omega) \to L^2(\Omega; b)$ by $I_\varepsilon = I_\varepsilon^{(3)} \circ I_\varepsilon^{(2)} \circ I_\varepsilon^{(1)}$:

$$H_0^1(\Omega) \xrightarrow{I_\varepsilon^{(1)}} H^1(\Omega_\varepsilon) \xrightarrow{I_\varepsilon^{(2)}} L^2(\Omega_\varepsilon) \xrightarrow{I_\varepsilon^{(3)}} L^2(\Omega; b)$$

where $I_\varepsilon^{(2)}$ is the injection map and where

$$I_\varepsilon^{(1)} w(x) = w(x) \qquad \text{for} \quad x \in \Omega_\epsilon,$$

and

$$\begin{aligned} I_\varepsilon^{(3)} w(x) &= w(x) \qquad && \text{if} \quad x \in \Omega_\varepsilon, \\ &= 0 && \text{if} \quad x \in \Omega \backslash \Omega_\varepsilon. \end{aligned}$$

Since, by Rellich's Lemma, $I_\varepsilon^{(2)}$ is compact and since $I_\varepsilon^{(1)}$ is continuous and $I_\varepsilon^{(3)}$ bounded, the map I_ε is compact. Hence the proof is complete if we show that

$$I_\varepsilon \to I \qquad \text{in the operator-norm as } \varepsilon \searrow 0, \tag{3.6}$$

where $I\colon H_0^1(\Omega) \to L^2(\Omega; b)$ is the injection map.

Let $w \in H_0^1(\Omega)$. Using Lemma 3.1 with $q = 2$, we find that

$$\begin{aligned} \|(I_\varepsilon - I)\, w\|^2_{L^2(\Omega; b)} &= \int_{\Omega \backslash \Omega_\varepsilon} b w^2 = \int_{\Omega \backslash \Omega_\varepsilon} b d_\Omega^2 \left(\frac{w}{d_\Omega} \right)^2 \\ &\leqslant C K(\varepsilon)\, \|w\|^2_{H_0^1(\Omega)} \end{aligned} \tag{3.7}$$

where

$$K(\varepsilon) = \max_{x \in \Omega \backslash \Omega_\varepsilon} b(x)\, d_\Omega^2(x).$$

By (3.4), $K(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ and thus (3.6) follows from (3.7).

The next question is what we can say about the existence of eigenvalues if b satisfies merely (3.3) instead of (3.4). The following counterexample shows that it may happen that Problem (3.1) has no eigenvalues at all.

COUNTEREXAMPLE. Consider the problem

$$\begin{aligned} &-w'' = (\lambda / x^2)\, w, \qquad x \in (0, 1), \\ &w(0) = w(1) = 0, \end{aligned}$$

i.e., $\Omega = (0, 1)$, $b = 1/x^2$ and $c \equiv 0$. This problem has the explicit solutions

$$-\sqrt{x} \log x \qquad \text{if} \quad \lambda = \tfrac{1}{4}$$

and

$$\sqrt{x} \sin\{\omega_\lambda \log x\} \qquad \text{if} \quad \lambda > \tfrac{1}{4}, \quad \omega_\lambda = \sqrt{(\lambda - \tfrac{1}{4})}.$$

Notice that these solutions do not belong to $H_0^1(0,1)$, i.e., they are generalized eigenfunctions and $[\frac{1}{4}, \infty)$ is a continuous spectrum. On the other hand an easy calculation shows that

$$\int_0^1 \left(\frac{w}{x}\right)^2 dx \leqslant 4 \int_0^1 w_x^2\, dx \qquad \text{for} \quad w \in H_0^1(0,1).$$

This implies that $\lambda \geqslant \frac{1}{4}$ for any λ in the spectrum. Hence this problem does not possess a point spectrum, i.e., there exist no eigenvalues.

This example implies that if b behaves like d_Ω^{-2} near the boundary $\partial\Omega$, we cannot always expect the existence of countably many eigenvalues λ_i with $\lambda_i \to \infty$ as $i \to \infty$. However it is still possible that a finite number of eigenvalues exist. The next theorem gives conditions on b and c such that the lowest eigenvalues exists. For this, suppose that $b \in L_{\text{loc}}^\infty(\Omega)$, $b(x) \geqslant b_0 > 0$ a.e. in Ω and let

$$H_0^1(\Omega; b) = \left\{ u \in H_0^1(\Omega) \colon \int_\Omega bu^2 < \infty \right\}.$$

Note that we do not impose any growth or integrability condition on b. The linear space $H_0^1(\Omega; b)$ is complete when equipped with the norm

$$\|u\|_{H_0^1(\Omega;b)}^2 = \int_\Omega |\nabla u|^2 + \int bu^2.$$

We define λ_1 and $\bar{\lambda}$ by

$$\lambda_1 = \inf_{\substack{w \in H_0^1(\Omega) \\ \int_\Omega bw^2 = 1}} \left\{ \int_\Omega |\nabla w|^2 - \int_\Omega cw^2 \right\} \tag{3.8}$$

and

$$\bar{\lambda} = \inf_{\substack{w \in H_0^1(\Omega) \\ \int_\Omega bw^2 = 1}} \int_\Omega |\nabla w|^2. \tag{3.9}$$

THEOREM 3.4. *Let b and c satisfy* (3.2). *If*

$$\limsup_{d_\Omega(x) \searrow 0} c(x)\, d_\Omega^2(x) \leqslant 0 \qquad (\textit{uniformly}) \tag{3.10}$$

and if $\lambda_1 < \bar{\lambda}$, then λ_1 is an eigenvalue of Problem (3.1). *The corresponding eigenfunction w_1 satisfies*

$$\lambda_1 = \frac{\int_\Omega |\nabla w_1|^2 - \int_\Omega cw_1^2}{\int_\Omega bw_1^2}. \tag{3.11}$$

Remark. In Section 5 we shall give an example where the conditions of Theorem 3.4 are satisfied (Example 5.2 and Remark 5.7).

Proof. Let $\{y_n\}$ be a minimizing sequence in $H_0^1(\Omega; b)$ for λ_1:

$$\int_\Omega |\nabla y_n|^2 - \int_\Omega c y_n^2 = \lambda_1 + o(1) \qquad \text{as} \quad n \to \infty,$$

and

$$\int_\Omega b y_n^2 = 1.$$

Since y_n is uniformly bounded in $H_0^1(\Omega; b)$, there exists a subsequence, which we denote by $\{y_n\}$ again, such that for any $\phi \in H_0^1(\Omega; b)$

$$\int_\Omega \nabla y_n \cdot \nabla \phi \to \int_\Omega \nabla y \cdot \nabla \phi \qquad \text{as} \quad n \to \infty, \tag{3.12}$$

and

$$\int_\Omega b y_n \phi \to \int_\Omega b y \phi \qquad \text{as} \quad n \to \infty, \tag{3.13}$$

for some $y \in H_0^1(\Omega; b)$. Since $|c| \leqslant K_1 b$, we also have

$$\int_\Omega c y_n \phi \to \int_\Omega c y \phi \qquad \text{as} \quad n \to \infty. \tag{3.14}$$

We claim that $w_1 = y$ satisfies (3.11) and is an eigenfunction corresponding to λ_1. Since b does not satisfy condition (3.4), we cannot apply Lemma 3.3 to conclude that

$$y_n \to y \qquad \text{in} \quad H_0^1(\Omega; b) \quad \text{as} \quad n \to \infty. \tag{3.15}$$

Instead we shall use a method due to Brézis and Nirenberg [8] to prove (3.15). Here the condition $\lambda_1 < \bar{\lambda}$ will compensate the lack of compactness.
Set $v_n = y_n - y$. Then, using (3.12), (3.13) and (3.14),

$$\begin{aligned} \int_\Omega |\nabla v_n|^2 + \int_\Omega |\nabla y|^2 - \int_\Omega c v_n^2 - \int_\Omega c y^2 &= \lambda_1 + o(1) \\ &= \lambda_1 \int_\Omega b y^2 + \lambda_1 \int_\Omega b v_n^2 + o(1) \qquad \text{as} \quad n \to \infty. \end{aligned}$$

By (3.8)

$$\int_\Omega |\nabla y|^2 - \int_\Omega cy^2 \geqslant \lambda_1 \int_\Omega by^2$$

and thus

$$\begin{aligned}\int_\Omega |\nabla v_n|^2 &\leqslant \int_\Omega cv_n^2 + \lambda_1 \int_\Omega bv_n^2 + o(1)\\ &< \int_\Omega c^+ v_n^2 + \bar{\lambda} \int_\Omega bv_n^2 + o(1) \qquad \text{as} \quad n \to \infty, \end{aligned} \tag{3.16}$$

because $\lambda_1 < \bar{\lambda}$. Here $c^+(x) = \max\{c(x), 0\}$. Since $v_n \to 0$ weakly in $H_0^1(\Omega)$ as $n \to \infty$, it follows from (3.10) and Lemma 3.3 that

$$\int_\Omega c^+ v_n^2 \to 0 \qquad \text{as} \quad n \to \infty.$$

Combined with (3.9) and (3.16) this yields that

$$\int_\Omega |\nabla v_n|^2 \to 0 \qquad \text{as} \quad n \to \infty.$$

This proves (3.15) and it follows at once that $w_1 = y$ satisfies (3.11).

Since w_1 minimizes (3.8), there exists a Lagrange multiplier λ such that

$$-\Delta w_1 = cw_1 + \lambda b w_1 \qquad \text{in} \quad H_0^1(\Omega; b).$$

By (3.11), $\lambda = \lambda_1$ and hence λ_1 is an eigenvalue of Problem (3.1). This completes the proof of Theorem 3.4.

Next we derive two results which we shall use in Section 4.

LEMMA 3.5. *Let $b, c \in C^\delta(\Omega)$ for some $\delta \in (0, 1)$. Let b and c satisfy* (3.2) *and let*

$$b(x) \leqslant C d_\Omega^{-1}(x), \qquad x \in \Omega. \tag{3.17}$$

Then the eigenfunctions $w_i(x)$ belong to $C^{2,\delta}(\Omega) \cap C^1(\bar{\Omega})$. In addition, $w_1 > 0$ (or < 0) in Ω and

$$\frac{\partial w_1}{\partial \nu} < 0 \qquad (or > 0) \ on \ \partial\Omega.$$

Proof. Since $b, c \in C^\delta(\Omega)$ we have $w_i \in C^{2,\delta}(\Omega)$. To prove $w_i \in C^1(\bar{\Omega})$, we observe that, by (3.17) and Lemma 3.1, $\Delta w_i \in L^2$, and hence

$$w_i \in W^{2,2}(\Omega).$$

However, we shall show that $w_i \in W^{2,p}(\Omega)$ for any $p \geqslant 1$, and thus prove that $w_i \in C^1(\bar{\Omega})$.

In [1, p. 216] it is proved that

$$W_0^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : \gamma_q(u) = 0 \text{ in } L^q(\partial\Omega)\},$$

where $\gamma_q : W^{1,q}(\Omega) \to L^q(\partial\Omega)$ is the trace operator. Thus

$$W^{1,q}(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1,q}(\Omega), \qquad 1 < p \leqslant q.$$

Since $w_i \in W^{2,2}(\Omega)$ we have $w_i \in W^{1,q}(\Omega)$ for $2 \leqslant q < 2N/(N-2)^+$ and we already know that $w_i \in W_0^{1,2}(\Omega)$. Hence

$$w_i \in W_0^{1,q}(\Omega) \qquad \text{for} \quad 2 \leqslant q \leqslant 2N/(N-2)^+.$$

By (3.17) and Lemma 3.1 we may now conclude that

$$w_i \in W^{2,q}(\Omega) \qquad \text{for} \quad 2 \leqslant q \leqslant 2N/(N-2)^+.$$

Iteration of the above argument yields that $w_i \in W^{2,p}(\Omega)$ for $p \geqslant 1$ and thus $w_i \in C^1(\bar{\Omega})$.

The proof of Lemma 3.5 is completed by (3.5) and the following boundary point lemma.

LEMMA 3.6. *Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy*

$$\begin{aligned} -\Delta w &\geqslant k(x)\, w \qquad && \text{in} \quad \Omega, \\ w &= 0 && \text{on} \quad \partial\Omega, \end{aligned}$$

where $k : \Omega \to \mathbb{R}$ satisfies

$$k(x) \geqslant -\rho(d_\Omega(x)), \qquad x \in \Omega,$$

for some $\rho \in C(\mathbb{R}^+)$ such that

$$\int_0^1 s\rho(s)\, ds < \infty.$$

If $w \geqslant 0$ in Ω and $w \not\equiv 0$, then

$$\frac{\partial w}{\partial \nu} < 0 \qquad \text{on} \quad \partial\Omega.$$

The proof of Lemma 3.6 follows the same lines as the proof of the standard boundary point lemma [19]. We omit it here.

LEMMA 3.7. *Let* $b_n, c_n, b, c \in L^1(\Omega) \cap C^\delta(\Omega)$ $(n = 1, 2, ...)$ *and let*

$$b_n \to b \quad \text{and} \quad c_n \to c \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty.$$

Let the pairs b_n, c_n *and* b, c *satisfy* (3.2) *and* (3.3) *with constants which do not depend on n. If* λ_{1n} *and* λ_1 *denote the first eigenvalue of Problem* (3.1) *related to* b_n, c_n *and* b, c, *then*

$$\lambda_{1n} \to \lambda_1 \quad \text{as} \quad n \to \infty.$$

Proof. Let w_{1n} and w_1 denote the related positive eigenfunctions with norm 1 in $C^1(\bar{\Omega})$. Then

$$\begin{aligned}\lambda_{1n} &\leqslant \frac{\int_\Omega |\nabla w_1|^2 - \int_\Omega c_n w_1^2}{\int_\Omega b_n w_1^2} \\ &= \frac{\int_\Omega |\nabla w_1|^2 - \int_\Omega c w_1^2 + \int_\Omega (c - c_n)\, w_1^2}{\int_\Omega b w_1^2 - \int_\Omega (b - b_n)\, w_1^2} \to \lambda_1 \qquad \text{as} \quad n \to \infty,\end{aligned}$$

and hence

$$\limsup_{n \to \infty} \lambda_{1n} \leqslant \lambda_1. \tag{3.18}$$

On the other hand

$$\lambda_1 \leqslant \frac{\int_\Omega |\nabla w_{1n}|^2 - \int_\Omega c w_{1n}^2}{\int_\Omega b w_{1n}^2} = \frac{\int_\Omega |\nabla w_{1n}|^2 - \int_\Omega c_n w_{1n}^2 - \int_\Omega (c - c_n)\, w_{1n}^2}{\int_\Omega b_n w_{1n}^2 + \int_\Omega (b - b_n)\, w_{1n}^2}$$

and hence

$$\lambda_1 \leqslant \liminf_{n \to \infty} \lambda_{1n} \tag{3.19}$$

provided

$$\liminf_{n \to \infty} \int_\Omega b_n w_{1n}^2 > 0. \tag{3.20}$$

Clearly (3.18) and (3.19) imply that $\lambda_{1n} \to \lambda_1$ as $n \to \infty$. Hence it remains to prove (3.20).

Using the fact that $|c_n| \leqslant K_1 b_n$ in Ω, we have

$$\lambda_{1n} = \frac{\int_\Omega |\nabla w_{1n}|^2 - \int_\Omega c_n w_{1n}^2}{\int_\Omega b_n w_{1n}^2} \geqslant -K_1.$$

In view of (3.18) this implies that λ_{1n} is uniformly bounded. Combined with (3.2), (3.17) and the fact that $\|w_{1n}\|_{C^1(\bar{\Omega})} = 1$, this yields that

$$\|\Delta w_{1n}\|_{L^\infty(\Omega)} \leqslant K, \qquad n = 1,2,...,$$

for some $K > 0$. Hence there exists a subsequence which we denote by $\{w_{1n}\}$ again, such that

$$w_{1n} \to \tilde{w} \quad \text{in} \quad C^1(\bar{\Omega}) \quad \text{as} \quad n \to \infty,$$

for some $\tilde{w} = C^1(\bar{\Omega})$ with norm 1 in $C^1(\bar{\Omega})$. Hence

$$\int_\Omega b_n w_{1n}^2 \geqslant b_0 \int_\Omega w_{1n}^2 \to b_0 \int_\Omega \tilde{w}^2 > 0 \qquad \text{as} \quad n \to \infty,$$

which proves (3.20).

4. The Principle of Linearized Stability

In this section we prove our main results.

Let $\bar{v} \in C^{2,\alpha}(\bar{\Omega})$ be a solution of the problem

$$\text{(II)} \begin{cases} \Delta \bar{v} + f(\bar{v}) = 0 & \text{in} \quad \Omega, \\ \bar{v} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

which satisfies (1.3). Then a formal linearization of Eq. (1.1) around $\bar{v}$ yields the eigenvalue problem

$$\text{(III)} \begin{cases} -\Delta w = f'(\bar{v})\, w + \lambda \beta'(\bar{v})\, w & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By Lemma 2.4

$$\beta'(s) \leqslant C s^{-1}, \qquad s \in (0, s_0], \tag{4.1}$$

where $C = s_0 \beta'(s_0)$, and since $\partial v/\partial \nu < 0$ on $\partial\Omega$,

$$\beta'(\bar{v}(x)) \leqslant K_1 \{d_\Omega(x)\}^{-1}, \qquad x \in \Omega,$$

for some $K_1 > 0$. Moreover, since $f \circ \beta^{-1}$ is Lipschitz continuous, we also have

$$|f'(\bar{v}(x))| \leqslant K_2 \{d_\Omega(x)\}^{-1}, \qquad x \in \Omega,$$

for some $K_2 > 0$. Hence Problem III falls into the class of eigenvalue problems which we discussed in Section 3, with $b = \beta'(\bar{v})$ and $c = f'(\bar{v})$. In particular, by Theorem 3.2, the first eigenvalue λ_1 of Problem III is well defined.

In the following two subsections we study the cases $\lambda_1 > 0$ and $\lambda_1 < 0$. Throughout this section v_0 denotes the initial function for Problem I and $v(x, t; v_0)$ is the corresponding solution.

Case 1: $\lambda_1 > 0$.

The following theorem shows that if $\lambda_1 > 0$ then $\bar{v}$ is exponentially stable in $L^\infty(\Omega)$.

THEOREM 4.1 (The Principle of Linearized Stability). *Let Ω, β, f and v_0 satisfy hypothesis* (H) *and let $\bar{v} \in C^{2,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ be a solution of Problem* II *which satisfies* (1.3). *Let $\lambda_1 > 0$ be the first eigenvalue of Problem* III. *Then there exist positive constants ε, C and T such that*

$$|v(x, t; v_0) - \bar{v}(x)| \leqslant Ce^{-\lambda_1 t} e(x), \qquad x \in \bar{\Omega}, \quad t \geqslant T,$$

for all initial functions $v_0 \geqslant 0$ satisfying

$$\|v_0 - \bar{v}\|_{L^\infty(\Omega)} \leqslant \varepsilon.$$

Recall that the function $e\colon \bar{\Omega} \to [0, \infty)$ is defined by (1.6).

Remark 4.2. Theorem 4.1 remains valid if we replace the boundary condition $u = 0$ on $\partial\Omega \times \mathbb{R}^+$ by $u = \chi(x) \geqslant 0$ on $\partial\Omega \times \mathbb{R}^+$, if we replace Δ by a more general self-adjoint uniformly elliptic operator L and if β and f depend explicitly on x (i.e., $\beta(x, u), f(x, u)$), provided the dependence on x is smooth enough and β and f satisfy hypotheses (H2) and (H3) uniformly with respect to x.

The proof of Theorem 4.1 is divided into three steps, which are developed below in the Propositions 4.3, 4.6 and 4.7. Loosely speaking, Proposition 4.3 asserts the existence of a family of arbitrarily small invariant neighbourhoods of $\bar{v}$ in $L^\infty(\Omega)$, thus proving the stability of $\bar{v}$ in $L^\infty(\Omega)$. Proposition 4.6 shows that for any initial function in such a sufficiently small invariant neighbourhood of $\bar{v}$, the normal derivative of $v(x, t; v_0) - \bar{v}(x)$ on $\partial\Omega$ vanishes as $t \to \infty$. Once $v(x, t; v_0)$ behaves "properly" near the lateral boundary, then in Proposition 4.7, the solution is "squeezed" between appropriately constructed sub- and supersolutions which converge exponentially to $\bar{v}$.

PROPOSITION 4.3. *Let the hypotheses of Theorem* 4.1 *be satisfied. For all sufficiently small $\varepsilon > 0$, there exist functions $\underline{\psi}_\varepsilon$, $\bar{\psi}_\varepsilon \in C(\bar{\Omega})$ such that*

(i) $0 \leqslant \underline{\psi}_\varepsilon(x) \leqslant [\bar{v}(x) - \delta_\varepsilon]^+$, $\quad \bar{\psi}_\varepsilon(x) \geqslant \bar{v}(x) + \delta_\varepsilon$, $\quad x \in \Omega$, *where* $[a]^+ = \max\{0, a\}$, *$\delta_\varepsilon > 0$ and $\delta_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$;*

(ii) $\bar{\psi}_\varepsilon - \underline{\psi}_\varepsilon \to 0$ *as* $\varepsilon \to 0$, *uniformly on* Ω;

(iii) *if* $v_0 \in I_\varepsilon$ *then* $v(x, t; v_0) \in I_\varepsilon$ *for all* $t \geqslant 0$, *where*

$$I_\varepsilon = \{v \in L^\infty(\Omega) \colon \underline{\psi}_\varepsilon \leqslant v \leqslant \bar{\psi}_\varepsilon \text{ a.e. in } \Omega\}. \tag{4.2}$$

COROLLARY 4.4. *If* $v_0 \geqslant 0$ *and* $\|v_0 - \bar{v}\|_{L^\infty(\Omega)} \leqslant \delta$ *for some sufficiently small* $\delta > 0$, *then* $v(x, t; v_0) \in I_\varepsilon$ *for all* $t \geqslant 0$ *and for some* $\varepsilon > 0$.

Proof of Proposition 4.3. By the well-known deformation lemma, there exists a C^∞-map $\eta \colon (-1, 1) \times \mathbb{R}^N \to \mathbb{R}^N$ such that $\eta(\varepsilon, \cdot) \colon \mathbb{R}^N \to \mathbb{R}^N$ is a diffeomorphism for $\varepsilon \in (-1, 1)$, $\eta(0, \cdot)$ is the identity-map and $\Omega_\varepsilon \equiv \eta(\varepsilon, \Omega)$ satisfies

$$\bar{\Omega}_{\varepsilon_1} \subset \Omega_{\varepsilon_2} \qquad \text{if} \quad -1 < \varepsilon_1 < \varepsilon_2 < 1.$$

Consider the problem

$$(\mathrm{II}_\varepsilon) \begin{cases} \Delta v + f(v) = 0 & \text{in} \quad \Omega_\varepsilon, \\ v = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases}$$

Setting $\tilde{v}_\varepsilon(x) = v(\eta(\varepsilon, x))$, $x \in \Omega$, then equivalently

$$(\mathrm{II}'_\varepsilon) \begin{cases} A(\varepsilon, \tilde{v}_\varepsilon) + f(\tilde{v}_\varepsilon) = 0 & \text{in} \quad \Omega, \\ \tilde{v}_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $A \colon (-1, 1) \times C_0^{2,\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$ is continuous and $A(0, \cdot) = \Delta$. Here we have set

$$C_0^{2,\alpha}(\bar{\Omega}) = \{v \in C^{2,\alpha}(\bar{\Omega}) \colon v = 0 \text{ on } \partial\Omega\}.$$

Now $\tilde{v}_0 = \bar{v}$ solves Problem II'_0, i.e.,

$$\mathscr{F}(0, \bar{v}) = 0,$$

where the map $\mathscr{F} \colon (-1, 1) \times C_0^{2,\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$ is defined by

$$\mathscr{F}(\varepsilon, v) = A(\varepsilon, v) + f(v).$$

To solve the equation $\mathscr{F}(\varepsilon, v) = 0$ around $(0, \bar{v})$, we want to apply the implicit function theorem. The condition to be checked is the invertibility of the map $\mathscr{F}_v(0, \bar{v}) \colon C_0^{2,\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$, which is given by

$$\mathscr{F}_v(w) = \Delta w + f'(\bar{v})\, w.$$

This is done in the following lemma.

LEMMA 4.5. *The map* $\mathscr{F}_v$ *is bijective and has a bounded inverse.*

Proof. The hypothesis $sf'(s) \in C^\alpha(\overline{\mathbb{R}^+})$ guarantees that the range of $\mathscr{F}_v$ is in $C^\alpha(\bar{\Omega})$. To show that $\mathscr{F}_v$ is bijective, let $h \in C^\alpha(\bar{\Omega})$ and consider the problem

$$\Delta w + f'(\bar{v})\, w = h \qquad \text{in} \quad H_0^1(\Omega). \tag{4.3}$$

Since $\lambda_1 > 0$, $\lambda = 0$ is not an eigenvalue of the problem

$$\Delta w + f'(\bar{v})\, w = \lambda \beta'(\bar{v}) w, \qquad w \in H_0^1(\Omega),$$

and it follows from the theory of Section 3 that (4.3) has a unique solution $w \in H_0^1(\Omega)$. As in the proof of Lemma 3.5 we conclude that w in fact belongs to $W^{1,p}(\Omega)$ for $1 \leqslant p < \infty$, and hence $w \in C^1(\bar{\Omega})$. By (H3) then $f'(\bar{v})w \in C^\alpha(\bar{\Omega})$, whence $w \in C_0^{2,\alpha}(\bar{\Omega})$. Therefore $\mathscr{F}_v$ is bijective.

It is easy to see that $\mathscr{F}_v$ is closed, and hence, by the Closed Graph Theorem, $\mathscr{F}_v^{-1}$ is continuous. This completes the proof of Lemma 4.5.

To finish the proof of Proposition 4.3, we apply the implicit function theorem to conclude that there exists a unique, continuous, one-parameter family of functions $\tilde{v}_\varepsilon$ in $C_0^{2,\alpha}(\bar{\Omega})$ solving Problem II'_ε for $|\varepsilon|$ small enough, such that $\tilde{v}_0 = \bar{v}$. Note that $\tilde{v}_\varepsilon \geqslant 0$ for all small $|\varepsilon|$, because $\tilde{v}_\varepsilon$ is near $\bar{v}$ in $C_0^{2,\alpha}(\bar{\Omega})$. We define, for small and positive ε,

$$\bar{\psi}_\varepsilon(x) = \tilde{v}_\varepsilon(\eta^{-1}(\varepsilon, x)), \qquad x \in \Omega,$$

and

$$\begin{aligned} \underline{\psi}_\varepsilon(x) &= v_{-\varepsilon}(\eta^{-1}(-\varepsilon, x)), \qquad & x \in \Omega_{-\varepsilon}, \\ &= 0, & x \in \bar{\Omega} \backslash \Omega_{-\varepsilon}. \end{aligned}$$

Then $\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ satisfy assertion (ii) of Proposition 4.3. Furthermore, since

$$\begin{aligned} \Delta \bar{\psi}_\varepsilon + f(\bar{\psi}_\varepsilon) &= 0 \qquad \text{in} \quad \Omega, \\ \bar{\psi}_\varepsilon &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} \Delta \underline{\psi}_\varepsilon + f(\underline{\psi}_\varepsilon) &= 0 \qquad \text{in} \quad \Omega_{-\varepsilon}, \\ \underline{\psi}_\varepsilon &= 0 \qquad \text{in} \quad \bar{\Omega} \backslash \Omega_{-\varepsilon}, \end{aligned}$$

$\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ are a super-, respectively, and a subsolution of Problem I. Assuming for the moment that, for sufficiently small ε,

$$\underline{\psi}_\varepsilon \leqslant \bar{v} \leqslant \bar{\psi}_\varepsilon \qquad \text{in} \quad \Omega, \tag{4.4}$$

assertion (i) follows easily from the strong maximum principle and the construction of $\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$. Finally, assertion (iii) is an immediate consequence of Proposition 2.3.

So it remains to prove (4.4), which essentially follows from the fact that $\lambda_1 > 0$. Here we shall prove that

$$\underline{\psi}_\varepsilon \leqslant \bar{v} \quad \text{in} \quad \Omega,$$

since the proof of the second inequality in (4.4) is similar. From the construction of $\underline{\psi}_\varepsilon$ and that $\tilde{v}_\varepsilon$ is near $\bar{v}$ in $C_0^{2,\alpha}(\bar{\Omega})$ it follows that

$$\underline{\psi}_\varepsilon(x) \leqslant \bar{v}(x) + \sigma(\varepsilon)\, e(x), \qquad x \in \Omega,$$

where $e(x)$ is defined by (1.6) and where $0 \leqslant \sigma(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$. Now the result follows at once from the fact that $\underline{\psi}_\varepsilon$ is a subsolution and Proposition 4.7, which we shall formulate later.

PROPOSITION 4.6. *Let the hypotheses of Theorem* 4.1 *be satisfied and let* I_ε *be defined by* (4.2) *with* $\varepsilon > 0$ *sufficiently small. Then for any* $\sigma > 0$ *there exists a* $T = T(\sigma) > 0$ *such that*

$$|v(x, t; v_0) - \bar{v}(x)| \leqslant \sigma e(x), \qquad x \in \Omega, \quad t \geqslant T(\sigma), \tag{4.5}$$

for all v_0 *in* I_ε. *Here the function* $e(x)$ *is defined by* (1.6).

Proof. As a first step we claim that there exist constants $\gamma \in (0, 1)$ and $T_1 > 0$ such that

$$\gamma e(x) \leqslant v(x, t; v_0) \leqslant \gamma^{-1} e(x), \qquad x \in \Omega, \quad t \geqslant T_1, \tag{4.6}$$

for all $v_0 \in I_\varepsilon$ with ε sufficiently small.

The first inequality in (4.6) is proved in the Appendix of the present paper. This lower bound is important since it enables us to handle the degeneracy of the equation in Problem I near points where $v = 0$. When $f(s) \geqslant -Ks$ for some $K > 0$, this "positivity results" holds for all $v_0 \geqslant 0$ ($v_0 \not\equiv 0$) [5] (see also[4]).

The second inequality in (4.6) is proved in [6, Appendix].

We define for small $\varepsilon > 0$

$$\underline{q}_\varepsilon(x) = \max\{\underline{\psi}_\varepsilon(x), \gamma e(x)\}, \qquad x \in \bar{\Omega},$$

and

$$\bar{q}_\varepsilon(x) = \min\{\bar{\psi}_\varepsilon(x), \gamma^{-1} e(x)\}, \qquad x \in \bar{\Omega}.$$

Then, by Proposition 4.3 and the inequalities (4.6), we have, for any v_0 in I_ε,

$$\underline{q}_\varepsilon(x) \leqslant v(x, t; v_0) \leqslant \bar{q}_\varepsilon(x), \qquad x \in \bar{\Omega}, \quad t \geqslant T_1. \tag{4.7}$$

Next, set $z_1(x, t) = v(x, t; v_0) - \bar{v}(x)$. Then z_1, which we denote by z, satisfies the equation

$$\mathscr{L}_1(z) \equiv -bz_t + \Delta + cz = 0 \qquad \text{in} \quad \Omega \times (T_1, \infty),$$

where

$$b(x, t) = \beta'(v(x, t; v_0))$$

and

$$c(x, t) = \int_0^1 f'(\mu v(x, t; v_0) + (1-\mu)\,\bar{v}(x))\,d\mu.$$

From (4.7) we obtain that for $x \in \Omega$ and $t \geqslant T_1$

$$b(x, t) \leqslant b_\varepsilon(x) \equiv \max_{\underline{q}_\varepsilon(x) \leqslant s \leqslant \bar{q}_\varepsilon(x)} \beta'(s) \tag{4.8}$$

and

$$c(x, t) \leqslant c_\varepsilon(x) \equiv \max_{\underline{q}_\varepsilon(x) \leqslant s \leqslant \bar{q}_\varepsilon(x)} \int_0^1 f'(\mu s + (1-\mu)\,\bar{v}(x))\,d\mu. \tag{4.9}$$

Since $b_\varepsilon(x) \geqslant \beta'(\gamma e(x))$ for x near $\partial\Omega$ and since $\partial e/\partial v < 0$ at $\partial\Omega$, Lemma 3.5 yields that the eigenvalue problem

$$-\Delta w = c_\varepsilon w + \lambda b_\varepsilon w, \qquad w \in H_0^1(\Omega), \tag{4.10}$$

possesses eigenvalues and classical eigenfunctions. Since $\beta'(d_\Omega(\cdot)) \in L^1(\Omega)$, it follows easily from the construction of $\underline{q}_\varepsilon$ and $\bar{q}_\varepsilon$ that

$$b_\varepsilon \to \beta'(v) \qquad \text{and} \qquad c_\varepsilon \to f'(\bar{v}) \qquad \text{in} \quad L^1(\Omega) \quad \text{as} \quad \varepsilon \searrow 0.$$

Thus, by Lemma 3.7, the first eigenvalue $\lambda_{1\varepsilon}$ of (4.10) satisfies

$$\lambda_{1\varepsilon} \to \lambda_1 \qquad \text{as} \quad \varepsilon \searrow 0.$$

In particular, since $\lambda_1 > 0$, $\lambda_{1\varepsilon} > 0$ for ε small enough.

We fix such $\varepsilon > 0$. In view of (4.7) we may choose a first eigenfunction w_ε of (4.10) such that

$$|v(x, T_1; v_0) - \bar{v}(x)| \leqslant e^{-\lambda_{1\varepsilon} T_1} w_\varepsilon(x), \qquad x \in \Omega.$$

Using (4.8) and (4.9), an easy computation yields that

$$\mathscr{L}_1(e^{-\lambda_{1\varepsilon} t} w_\varepsilon(x)) \leqslant 0 \qquad \text{in} \quad \Omega(T_1, \infty).$$

Hence

$$|v(x, t; v_0) - \bar{v}(x)| \leqslant e^{-\lambda_{1\varepsilon} t} w_\varepsilon(x), \qquad x \in \bar{\Omega}, \quad t \geqslant T_1. \tag{4.11}$$

Since $\partial w_\varepsilon/\partial \nu < 0$ on $\partial\Omega$, it follows from (4.11) that for some $T = T(\sigma)$ (4.5) holds.

PROPOSITION 4.7. *Let the hypotheses of Theorem* 4.1 *be satisfied. Then there exist positive constants* σ_0 *and* C *such that*

$$|v(x, t; v_0) - \bar{v}(x)| \leqslant Ce^{-\lambda_1 t} e(x), \qquad x \in \bar{\Omega}, \quad t \geqslant 0, \tag{4.12}$$

for any initial function v_0 *which satisfies*

$$|v_0(x) - \bar{v}(x)| \leqslant \sigma_0 e(x), \qquad x \in \bar{\Omega}.$$

Proof. Let w be a positive eigenfunction of Problem III corresponding to the first eigenvalue λ_1. Then, by Lemma 3.5, $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and it satisfies

$$-\Delta w = f'(\bar{v})\, w + \lambda_1 \beta'(\bar{v})\, w \qquad \text{in} \quad \Omega,$$

$$w = 0 \qquad \text{and} \qquad \partial w/\partial \nu < 0 \qquad \text{on} \quad \partial\Omega, \quad w > 0 \quad \text{in} \quad \Omega.$$

We proceed to construct super- and subsolutions for Problem I in the form $z_2(x, t) = \bar{v}(x) + g_1(t)\, w(x)$ and $z_3(x, t) = \bar{v}(x) - g_2(t)\, w(x)$, respectively, where $g_i(t) > 0$ for $t \geqslant 0$ and $g_i(t) = O(e^{-\lambda_1 t})$ as $t \to \infty$ for $i = 1, 2$.

We describe here the construction of the function g_2, the computation for g_1 being similar. We set $z \equiv z_3$ and $g \equiv g_2$. We assume that

$$\bar{v} - gw \geqslant \tfrac{1}{2}\bar{v} \qquad \text{in} \quad \Omega \times \mathbb{R}^+. \tag{4.13}$$

Then

$$\begin{aligned}\mathscr{L}_2(z) &\equiv -\beta(z)_t + \Delta z + f(z)\\ &= w\left\{\beta'(z)\, g' + \frac{f(\bar{v} - gw) - f(\bar{v})}{w} + gf'(\bar{v}) + \lambda_1 g\beta'(\bar{v})\right\}.\end{aligned}$$

By the mean value theorem, there exists a function $\xi\colon \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$0 < \xi(x, t) < w(x), \qquad x \in \Omega, \quad t > 0,$$

and such that

$$\begin{aligned}\mathscr{L}_2(z) &= w\{\beta'(z)\, g' + \tfrac{1}{2} f''(\bar{v} - g\xi)\, g^2 w + \lambda_1 g\beta'(\bar{v})\}\\ &= w\beta'(z)\left\{g' + \lambda_1 g\, \frac{\beta'(\bar{v})}{\beta'(\bar{v} - gw)} + \tfrac{1}{2} g^2 \frac{f''(\bar{v} - g\xi)\, w}{\beta'(\bar{v} - gw)}\right\}.\end{aligned} \tag{4.14}$$

We shall estimate the fractional quantities in the above expression. The delicate area is the neighbourhood of the boundary $\partial\Omega \times \mathbb{R}^+$, where $\bar{v}$ and $\bar{v}-gw$ are small. So suppose that x is such that $\bar{v}(x) \leqslant s_0$. Then we conclude from Lemma 2.4 that

$$\frac{\beta'(\bar{v})}{\beta'(\bar{v}-gw)} \geqslant \frac{\bar{v}-gw}{\bar{v}} \qquad \text{for} \quad x \text{ near } \partial\Omega.$$

Hence, using (4.13),

$$\frac{\beta'(\bar{v})}{\beta'(\bar{v}-gw)} \geqslant 1 - A_1 g \qquad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{4.15}$$

for some $A_1 > 0$.

From (H3) and (4.13) we find that

$$\frac{|f''(\bar{v}-g\xi)|}{\beta'(\bar{v}-g\xi)} \leqslant \frac{K}{\bar{v}-g\xi} \leqslant \frac{2K}{\bar{v}} \qquad \text{for} \quad x \text{ near } \partial\Omega,$$

and hence, since $\xi < w$ and $w \leqslant B\bar{v}$ for some $B > 0$,

$$\frac{|f''(\bar{v}-g\xi)|\, w}{\beta'(\bar{v}-gw)} \leqslant 2KB \qquad \text{for} \quad x \text{ near } \partial\Omega.$$

Thus we have for some $A_2 > 0$

$$\frac{|f''(\bar{v}-g\xi)|\, w}{\beta'(\bar{v}-gw)} \leqslant A_2 \qquad \text{in} \quad \Omega \times \mathbb{R}^+. \tag{4.16}$$

Using the inequalities (4.15) and (4.16), we obtain from (4.14) that

$$\mathscr{L}_2(z) \geqslant w\beta'(z)\{g' + \lambda_1 g(1 - A_3 g)\} \qquad \text{in} \quad \Omega \times \mathbb{R}^+,$$

where $A_3 = A_1 + \frac{1}{2}A_2/\lambda_1$. We define g as the solution of the problem

$$g' + \lambda_1 g(1 - A_3 g) = 0, \qquad t > 0,$$
$$g(0) = A_4 > 0,$$

where A_4 is so small that $1 - A_3 A_4 > 0$ and $\bar{v} - A_4 w \geqslant \frac{1}{2}\bar{v}$ in Ω. Then (4.13) is satisfied,

$$\mathscr{L}_2(z) \geqslant 0 \qquad \text{in} \quad \Omega \times \mathbb{R}^+,$$

and $g(t) = O(e^{-\lambda_1 t})$ as $t \to \infty$.

To complete the proof of Proposition 4.7 we choose $\sigma_0 > 0$ such that

$$\sigma_0 e(x) \leqslant A_4 w(x), \qquad x \in \Omega.$$

Proof of Theorem 4.1. The proof follows at once from applying successively the Propositions 4.3, 4.6 and 4.7.

Case 2: $\lambda_1 < 0$.
We shall prove the following theorem, which establishes the instability of $\bar{v}$ when $\lambda_1 < 0$.

THEOREM 4.8. *Let Ω, β, f and v_0 satisfy hypothesis* (H) *and let $v \in C^{2,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ be a solution of Problem* II *which satisfies* (1.3). *Let $\lambda_1 < 0$ be the first eigenvalue of Problem* III. *Then there exist a number $\tilde{\varepsilon} > 0$ and, for any $0 \leqslant v_0 \leqslant \bar{v}$ (respectively, $v_0 \geqslant \bar{v}$) with $v_0 \not\equiv \bar{v}$, a number $T = T(v_0) \geqslant 0$, such that*

$$v(x, t; v_0) < \bar{v}(x) - \tilde{\varepsilon} e(x), \qquad x \in \Omega, \quad t \geqslant T, \tag{4.17}$$

respectively,

$$v(x, t; v_0) > \bar{v}(x) + \tilde{\varepsilon} e(x), \qquad x \in \Omega, \quad t \geqslant T. \tag{4.18}$$

Remark 4.9. Again this theorem can be generalized to the cases mentioned in Remark 4.2.

We divide the proof of Theorem 4.8 into two steps. In Proposition 4.10 we apply a variant of the strong maximum principle to bound the quantity $v(x, t; v_0) - \bar{v}(x)$ away from 0 for $t > 0$. Then we construct in Proposition 4.12 suitable sub- and supersolutions to obtain (4.17) and (4.18).

PROPOSITION 4.10. *Let the hypotheses of Theorem* 4.8 *be satisfied. Then, for any $\tau > 0$, there exists a constant $\sigma_1 = \sigma_1(\tau; v_0) > 0$ such that*

$$|v(x, \tau; v_0) - \bar{v}(x)| \geqslant \sigma_1 e(x), \qquad x \in \Omega.$$

The main tool in the proof is the following lemma, which is a variant of the strong maximum principle.

LEMMA 4.11. *Let $v(x, t; v_{01})$ and $v(x, t; v_{02})$ be solutions of Problem* I *in $\bar{\Omega} \times [0, T]$, let*

$$v_{01} \leqslant v_{02} \qquad \text{and} \qquad v_{01} \not\equiv v_{02} \qquad \text{in } \Omega,$$

and let

$$v(x, t; v_{01}) \geqslant \kappa e(x), \qquad x \in \Omega, \quad 0 \leqslant t \leqslant T, \tag{4.19}$$

for some $\kappa > 0$. Then there exists a function $s \in C([0, T])$ such that $s(t) > 0$ for $0 < t \leqslant T$ and

$$v(x, t; v_{02}) - v(x, t; v_{01}) \geqslant s(t)\, e(x), \qquad x \in \Omega, \quad 0 \leqslant t \leqslant T. \tag{4.20}$$

Proof. By (4.19), $v(x, t; v_{0i}) \in C(\bar{\Omega} \times [0, T]) \cap C^{2,1}(\Omega \times (0, T])$ for $i = 1, 2$, and it follows from the strong maximum principle that (4.20) holds in any compact subset of Ω.

To prove (4.20) near $\partial\Omega$, we introduce the function $z_4(x, t) = v(x, t; v_{02}) - v(x, t; v_{01})$. Then $z \equiv z_4$ satisfies the linear equation

$$a(x, t)\, z_t = \Delta z + c(x, t)\, z$$

for some functions a and c. In particular

$$|c(x, t)| \leqslant Ka(x, t) \qquad \text{in} \quad \Omega \times [0, T], \tag{4.21}$$

for some $K > 0$, and, by (4.19),

$$a(x, t) \leqslant \tilde{k}(d_\Omega(x)) \qquad \text{with} \quad \int_0^1 \tilde{k}(s)\, ds < \infty, \tag{4.22}$$

for some function $\tilde{k}$ (here we may choose $\tilde{k}(s) = \beta'(Cs)$ for some $C > 0$).

As in the proof of the corresponding result where $a(x, t)$ is bounded [19], a barrier function is constructed to prove (4.20) near $\partial\Omega$. This is made possible by the properties (4.21) and (4.22). We leave the details to the reader.

Proof of Proposition 4.10. When $v_0 \geqslant \bar{v}$, $v_0 \not\equiv \bar{v}$ in Ω, then Proposition 4.10 follows at once from Lemma 4.11.

Next let $0 \leqslant v_0 \leqslant \bar{v}$ and $v_0 \not\equiv \bar{v}$ in Ω. We fix $\tau > 0$. Let w denote again a positive eigenfunction of Problem III corresponding to λ_1. We may assume without loss of generality that

$$v_0 \geqslant \bar{v} - \sigma_2 e^{-\lambda\tau} w \qquad \text{in } \Omega, \tag{4.23}$$

for some small $\sigma_2 > 0$ and for some $\lambda > |\lambda_1|$.

We want to apply Lemma 4.11. The condition to be checked is (4.19).

Consider the function

$$z_5(x, t) = \bar{v}(x) - g(t)\, w(x), \qquad g(t) = \sigma_2 e^{\lambda(t - \tau)}.$$

Then, writing $z = z_5$ again,

$$\begin{aligned} \mathscr{L}_2(z) &= -\beta(z)_t + \Delta z + f(z) \\ &= gw\{\lambda\beta'(\bar{v} - gw) + \lambda_1 \beta'(\bar{v}) + \tfrac{1}{2} gwf''(v - g\xi)\} \qquad \text{in} \quad \Omega \times (0, \tau], \end{aligned} \tag{4.24}$$

where $0 < \xi(x, t) < w(x)$ in $\Omega \times (0, \tau]$ and $\bar{v} - \sigma_2 w \geqslant \frac{1}{2}\bar{v}$ in Ω. By choosing σ_2 sufficiently small we find, using the hypothesis (H2) on β, that

$$\lambda\beta'(z) + \lambda_1 \beta'(\bar{v}) \geqslant K\beta'(\bar{v} - gw)$$

for some $K > 0$. By (4.16), we have that

$$\tfrac{1}{2} gwf''(\bar{v} - g\xi) \geqslant -\tfrac{1}{2} A_2 g\beta'(\bar{v} - gw).$$

Combining this with (4.24), we conclude that

$$\mathscr{L}_2(\bar{v} - gw) \geqslant 0 \qquad \text{in} \quad \Omega \times (0, \tau],$$

if we choose σ_2 sufficiently small. Using (4.23) this implies that

$$v(\cdot, t; v_0) \geqslant \bar{v} - \sigma_2 w \qquad \text{in} \quad \Omega, \quad t \in [0, \tau].$$

Hence condition (4.19) of Lemma 4.11 is satisfied and Proposition 4.10 follows at once.

PROPOSITION 4.12. *Let the hypotheses of Theorem* 4.8 *be satisfied and let for some* $\sigma_1 > 0$

$$0 \leqslant v_0 \leqslant \bar{v}(x) - \sigma_1 e(x), \qquad x \in \Omega,$$

respectively,

$$v_0 \geqslant \bar{v}(x) + \sigma_1 e(x), \qquad x \in \Omega.$$

Then there exist numbers $\tilde{\varepsilon} > 0$ *and* $T = T(\sigma_1) \geqslant 0$ *such that* (4.17), *respectively,* (4.18) *are valid.*

Proof. We shall only prove (4.17). The proof of (4.18) is similar.

Let $0 < \lambda < -\lambda_1$, let σ_3 and T be positive constants to be chosen later. Consider the function

$$z_6(x, t) = \bar{v}(x) - \sigma_3 e^{\lambda(t - T)} w(x), \qquad x \in \Omega, \quad 0 < t \leqslant T.$$

Using similar arguments as in the proof of Proposition 4.10, it follows from the fact that $\lambda_1 + \lambda < 0$ that

$$\mathscr{L}_2(z_6) \leqslant 0 \qquad \text{in} \quad \Omega \times (0, T],$$

if we choose σ_3 small enough. Hence, if we choose T so large that

$$\sigma_3 e^{-\lambda T} w(x) \leqslant \sigma_1 e(x), \qquad x \in \Omega,$$

it follows that

$$v(x, T; v_0) \leqslant \bar{v}(x) - \sigma_3 w(x), \qquad x \in \Omega.$$

Since

$$\mathscr{L}_2(\bar{v} - \sigma_3 w) \leqslant 0 \qquad \text{in} \quad \Omega \times (T, \infty),$$

we find that

$$v(x, t; v_0) \leqslant \bar{v}(x) - \sigma_3 w(x), \qquad x \in \Omega, \quad t \geqslant T_3,$$

and (4.17) follows at once.

Proof of Theorem 4.8. The proof follows at once from the Propositions (4.10) and (4.12).

5. An Application and Some Remarks

First we shal give an application of Theorem 4.1 Consider the problem

$$\text{(V)} \begin{cases} u_t = \Delta(u^m) + u^p & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases}$$

where $1 \leqslant p < m$. It is known that Problem V possesses a unique positive steady-state solution $\bar{u}$, and if $u_0 \geqslant 0$ and $u_0 \not\equiv 0$ in Ω, then the solution $u(x, t; u_0)$ of Problem V satisfies

$$u(x, t; u_0) \to \bar{u}(x) \qquad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad t \to \infty \tag{5.1}$$

(see Sacks [20]). If $p = 1$, this result can be improved to

$$|u(x, t; u_0) - \bar{u}(x)| \leqslant Ce^{-(m-1)t}\bar{u}(x), \qquad x \in \Omega, \quad t \geqslant T, \tag{5.2}$$

where C and T are positive constants which depend on u_0 (see Aronson and Peletier [3]).

We shall generalize (5.2) to the case $1 < p < m$. Let $\beta(s) = s^{1/m}$, $f(s) = s^{p/m}$ and $\bar{v} = \bar{u}^m$. Then $\bar{v}$ satisfies (1.3) and it follows from the concavity of the function f that the first eigenvalue λ_1 of Problem III is positive (the proof is similar to the proof in [15] of the fact that convexity of f (when $f(0) = 0$) implies that $\lambda_1 < 0$). Hence it follows from (5.1) and Theorem 4.1 that, if $1 \leqslant p < m$,

$$|u^m(x, t; u_0) - \bar{u}^m(x)| \leqslant Ce^{-\lambda_1 t}e(x), \qquad x \in \Omega, \quad t \geqslant T, \tag{5.3}$$

where C and T depend on u_0. In particular an explicit calculation shows that if $p = 1$, then $w = \bar{u}^m$ is an eigenfunction of Problem III with eigenvalue $\lambda_1 = m - 1$. Hence (5.2) is a special case of (5.3).

Next we make some remarks about equilibrium solutions $\bar{v}$ which are nonnegative in Ω, but either not everywhere positive or have zero slope at

$\partial\Omega$. By means of two examples we shall show the difficulties involved in such cases when applying the linearization methods of Section 4.

EXAMPLE 5.1. Let $m>1$ and $L\geqslant 0$. We consider the problem

$$\text{(VI)}\quad \begin{cases} u_t=(u^m)_{xx}-c_m u & \text{in}\quad (-L,1)\times\mathbb{R}^+,\\ u(-L,t)=0;\quad u(1,t)=1 & \text{for}\quad t>0,\\ u(\cdot,0)=u_0 & \text{in}\quad (-L,1),\end{cases}$$

where $c_m=2m(2m+1)/(m-1)^2$. Then Problem VI possesses a unique non-negative steady-state solution $\bar{u}$, given by

$$\begin{aligned}\bar{u}(x)&=0 &&\text{for}\quad x\in[-L,0],\\ &=x^\alpha &&\text{for}\quad x\in[0,1],\end{aligned}$$

where $\alpha=2/(m-1)$. It is easy to show that if $u_0\geqslant 0$ in $(-L,1)$, then the solution $u(x,t;u_0)$ of Problem VI satisfies

$$u(\cdot,t;u_0)\to\bar{u}\qquad\text{in}\quad L^\infty(\Omega)\quad\text{as}\quad t\to\infty. \tag{5.4}$$

We consider the question whether the convergence in (5.4) is exponential and whether linearization helps to answer this question.

First we consider the case $L=0$, i.e., $\bar{u}>0$ in $(-L,1)$. Then we arrive at the eigenvalue problem

$$\begin{aligned}&-w''=(\lambda-c_m)/(mx)^2\,w\qquad\text{in}\quad(0,1),\\ &w(0)=w(1)=0.\end{aligned}$$

We studied this problem in Section 3. In particular $w_1(x)=-\sqrt{x}\log x$ is a solution of this problem with $\lambda=\lambda_1=c_m+(m/4)$. We can use w_1 to construct a comparison function of the type $\bar{u}^m(x)+Ce^{-\lambda_1 t}w_1(x)$. Then it follows that for some $C>0$

$$u^m(x,t;u_0)-\bar{u}^m\leqslant Ce^{-\lambda_1 t}w_1(x),\qquad 0\leqslant x\leqslant 1,\quad t\geqslant 1. \tag{5.5}$$

However, the construction of a comparison function of the type $\bar{u}^m(x)-Ce^{-\lambda_1 t}w_1(x)$ fails, since for any $\varepsilon>0$ there exist values of $x\in(0,1)$ where $\bar{u}^m(x)-\varepsilon w_1(x)<0$.

If $L>0$, the situation is even worse. In that case linearization does not yield an estimate like (5.5).

Remark. Observe that in this example we were only able to give estimate (5.5) because we could explicitly solve $w_1(x)$. The existence of w_1 does not follow here from the theory of Section 3.

EXAMPLE 5.2. In [2] Aronson, Crandall and Peletier studied the problem

$$\text{(VII)}\quad \begin{cases} u_t=(u^m)_{xx}+u(1-u)(u-\alpha) & \text{in } (-L,L)\times\mathbb{R}^+, \\ u(\pm L,t)=0 & \text{for } t>0, \\ u(\cdot,0)=u_0 & \text{in } (-L,L), \end{cases}$$

where $0\leqslant u_0\leqslant 1$, $m>1$ and $0<\alpha<(m+1)/(m+3)$. In particular they show that there exists a positive number L_1 such that for $L>L_1$ Problem-VII possesses steady-state solutions with compact support contained in $(-L,L)$ and with values in $[0,1]$. Let $\bar{u}$ be such a steady-state solution, i.e., $\bar{u}$ satisfies

$$\begin{aligned} (\bar{u}^m)''+\bar{u}(1-\bar{u})(\bar{u}+\alpha)&=0 && \text{in } (-L,L), \\ \bar{u}&>0 && \text{in } (a,b), \\ \bar{u}_0&=0 && \text{in } [-L,a] \text{ and } [b,+L], \end{aligned}$$

where $-L<a<b<+L$.

Let

$$\beta(s)=s^{1/m},\qquad f(s)=s^{1/m}(1-s^{1/m})(s^{1/m}-\alpha),$$

and set $\bar{v}=\bar{u}^m$. Then an easy calculation shows that there exist positive constants C_i $(i=1,2,3)$ such that

$$C_1\{d_{(a,b)}(x)\}^{-2}\leqslant\beta'(\bar{v}(x))\leqslant C_2\{d_{(a,b)}(x)\}^{-2},\qquad x\in(a,b),$$

and

$$|f'(\bar{v}(x))|\leqslant C_3\beta'(\bar{v}(x)),\qquad x\in(a,b).$$

From the theory of Section 3 it follows that

$$\lambda_1=\inf_{w\in H_0^1(a,b)}\frac{\int_a^b\{(w')^2-f'(\bar{v})\,w^2\}}{\int_a^b\beta'(\bar{v})\,w^2} \tag{5.6}$$

exists. Since $\bar{v}'\in H_0^1(a,b)$ and since

$$-(\bar{v})''=f'(\bar{v})\,\bar{v}'\qquad\text{in } (a,b),$$

we know that $\lambda_1\leqslant 0$. Thus we may apply Theorem 3.4, and Problem III, restricted to (a,b), has a positive eigenfunction w_1 with eigenvalue $\lambda_1<0$ ($\lambda_1\neq 0$ since the eigenfunction $\bar{v}'$ with eigenvalue 0 changes sign). Hence we expect $\bar{u}$ to be unstable in some sense. This instability is made precise in the following result.

LEMMA 5.3. *Let $\bar{u}$ be a steady-state solution of Problem* VII *with compact support $[a, b]$. Let $0 \leqslant u_0 \leqslant 1$ be an initial function which satisfies*

$$u_0 \geqslant \bar{u} \qquad (\textit{resp.} \leqslant \bar{u}) \quad \textit{in} \quad (-L, L),$$

and

$$u_0 \not\equiv \bar{u} \qquad \textit{in} \quad (a, b). \tag{5.7}$$

Then there exist functions $p^+, p^- \in C([-L, L])$ with

$$p^{\pm} \equiv 0 \qquad \textit{in} \quad [-L, a] \cup [b, L],$$

and

$$0 < p^- < \bar{u} < p^+ \qquad \textit{in} \quad (a, b),$$

such that for some $T = T(u_0) \geqslant 0$

$$u(x, t; u_0) \geqslant p^+(x) \qquad \textit{for} \quad x \in [-L, L], \quad t \geqslant T,$$

respectively,

$$u(x, t; u_0) \leqslant p^-(x) \qquad \textit{for} \quad x \in [-L, L], \quad t \geqslant T,$$

where $u(x, t; u_0)$ is the solution of Problem VII.

Remark 5.4. Condition (5.7) is necessary. Indeed, if $L > 2L_1$, it is not difficult to construct an initial function u_0 which satisfies $u_0 \geqslant \bar{u}$ and $u_0 \not\equiv \bar{u}$ in $(-L, L)$, and $u_0 \equiv \bar{u}$ in (a, b), such that $u(x, t; u_0) \to \bar{u}$ as $t \to \infty$.

Remark 5.5. Although the precise statement in Lemma 5.3 is new, it is not difficult to describe the instability of $\bar{u}$ with the help of comparison functions (see [2]). The reason that we treat this example is only to illustrate the method of linearization.

Remark 5.6. Using results of [2], it is easy to show that the solution $u(x, t; u_0)$ in Lemma 5.3 satisfies $u(x, t; u_0) \to q(x)$ (resp. $u(x, t; u_0) \to 0$) as $t \to \infty$ where q is the maximal steady-state solution of Problem VII.

Unfortunately the eigenfunction w_1 is not suitable to construct comparison functions to prove Lemma 5.3. Instead we consider for small $\delta > 0$ the eigenvalue problem

$$\begin{aligned} -w'' &= f'(\bar{v})\, w + \lambda \beta'(\bar{v})\, w \qquad \text{in} \quad (a + \delta, b - \delta), \\ w(a + \delta) &= w(b - \delta) = 0. \end{aligned}$$

Let $\lambda_{1\delta}$ and $w_{1\delta}$ denote the first eigenvalue and a corresponding positive eigenfunction of this problem. Since $\lambda_1 < 0$, there exists a $\delta_0 > 0$ such that $\lambda_{1\delta} < 0$ for $\delta \leqslant \delta_0$. The proof of Lemma 5.3 follows easily (like the proof of Theorem 4.8) from the construction of comparison functions of the type

$$\begin{aligned} z_\delta(x, t) &= \bar{u} \pm \varepsilon e^{\mu t} w_{1\delta} && \text{in} \quad (a+\delta, b-\delta) \times [0, T], \\ &= \bar{u} && \text{in} \quad [-L, a+\delta] \cup [b-\delta, L] \times [0, T], \end{aligned}$$

where $\mu \in (0, |\lambda_{1\delta}|)$. We omit the details here.

Remark 5.7. If we replace the nonlinear function

$$f(s) = s^{1/m}(1 - s^{1/m})(s^{1/m} - \alpha)$$

which arises in Example 5.2 by the function

$$f_p(s) = s^{p/m}(1 - s^{p/m})(s^{p/m} - \alpha)$$

for some $p \in (1, m)$, then there exist again, for $L > 0$ big enough, steady-state solutions $\bar{v} = \bar{u}^m$ with compact support and the above analysis to show the instability of $\bar{v}$ goes through. Observe that in this case

$$C_1\{d_{(a,b)}(x)\}^{-\gamma} \leqslant \beta'(\bar{v}(x)) \leqslant C_2\{d_{(a,b)}(x)\}^{-\gamma}, \qquad x \in (a, b),$$

with

$$\gamma = 2(m-1)/(m-p) > 2,$$

i.e., $\beta'(\bar{v})$ does not satisfy the growth condition (3.3).

APPENDIX: A POSITIVITY RESULT

Here we shall prove that there exist numbers $\lambda > 0$, $T_1 > 0$ and $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$

$$v(x, t; v_0) \geqslant \gamma e(x), \qquad x \in \Omega, \tag{A.1}$$

for any $v_0 \in I_\varepsilon$, where I_ε is defined by (4.2).

The heart of the proof is the observation that if the initial function is a subsolution where it is positive and has negative slope at the boundary of the support, then the support is strictly expanding for small $t > 0$. In the next lemma this is proved for spherical symmetric initial functions.

Let $r_1 > r_0 > 0$ and let $p \in C^2(r_0, r_1) \cap C([r_0, r_1 + 1])$ satisfy

$$(\mathrm{A}_0)\quad \begin{cases} p'' + \dfrac{N-1}{r} p' + f(p) = 0, \quad p' \leqslant 0, \quad p > 0 & \text{on} \quad (r_0, r_1), \\ p(r) = 0 & \text{for} \quad r_1 \leqslant r \leqslant r_1 + 1, \\ \lim\limits_{r \nearrow r_1} p'(r) < 0 \text{ and } 0 < p(r_0) \leqslant s_0. \end{cases}$$

Such p exists for $r_1 - r_0$ small enough. For any $0 < \varepsilon \leqslant 1$ we define $v_\varepsilon(r, t)$ to be the solution of the problem

$$(\mathrm{P}_\varepsilon)\quad \begin{cases} \beta(v)_t = v'' + \dfrac{N-1}{r} v' + f(v), & r_0 < r < r_1 + \varepsilon, \quad t > 0, \\ v(r_0, t) = p(r_0) \quad \text{and} \quad v(r_1 + \varepsilon, t) = 0 & t > 0, \\ v(r, 0) = p(r), & r_0 < r < r_1 + \varepsilon. \end{cases}$$

LEMMA A. *Let β and f satisfy* (H2) *and* (H3). *Then, for $\varepsilon > 0$ small enough, there exist positive constants k and T such that*

$$v_\varepsilon(r, T) > k(r_1 + \varepsilon - r), \qquad r_0 \leqslant r \leqslant r_1 + \varepsilon.$$

Proof. First we consider the case $\varepsilon = 1$. Since $p(r)$ is a subsolution and not a solution of Problem P_1, $v_1(r, t)$ is nondecreasing with respect to t, and

$$v_1(\cdot, t) \not\equiv p \qquad \text{on} \quad (r_0, r_1 + 1) \qquad \text{for} \quad t > 0. \tag{A.2}$$

Thus $v_1(r_1, t) > 0$ when $t > 0$, for if $v_1(r_1, t_0) = 0$ for some $t_0 > 0$, then $v_1(r_1, t) = 0$ for $0 \leqslant t \leqslant t_0$ and $v_1(\cdot, t) \equiv p$ on $(r_0, r_1 + 1)$ for $0 \leqslant t \leqslant t_0$ which contradicts (A.2). Since $v_1 \in C([r_0, r_1 + 1] \times [0, \infty))$ [9], there exists a positive constant $\varepsilon < 1$ such that

$$v_1(r_1 + \varepsilon, 1) > 0. \tag{A.3}$$

We claim that the solution v_ε of Problem P_ε satisfies

$$v_\varepsilon(r, 1) > 0 \qquad \text{for} \quad r_0 < r < r_1 + \varepsilon. \tag{A.4}$$

Suppose that (A.4) is not true. Since v_ε is nondecreasing with respect to t and r, there exists a number $r_2 < r_1 + \varepsilon$ such that

$$v_\varepsilon(r, t) = 0, \qquad r_2 \leqslant r \leqslant r_1 + \varepsilon, \qquad 0 \leqslant t \leqslant 1.$$

Then clearly v_ε, extended to $(r_1 + \varepsilon, \infty) \times [0, 1]$ by 0, is a solution of Problem P_1. This contradicts (A.3), whence (A.4) holds.

Thus there exists a $t^* \in [0, 1]$ such that

$$v_\varepsilon(r, t) > 0, \qquad r_0 \leqslant r < r_1 + \varepsilon, \qquad t > t^*,$$

and

$$v_\varepsilon(r, t) \not> 0, \qquad r_0 \leqslant r < r_1 + \varepsilon, \quad 0 \leqslant t \leqslant t^*. \tag{A.5}$$

Then v_ε is a classical solution of Problem P_ε for $t > t^*$ and

$$v_{\varepsilon r}(\cdot, t) \in C(r_0, r_1 + \varepsilon), \qquad t > t^*.$$

Since $v_{\varepsilon t} \geqslant 0$, $v_{\varepsilon rr}$ is bounded from below on $(r_0, r_1 + \varepsilon) \times (t^*, \infty)$. Combined with the fact that $v_{\varepsilon r} \leqslant 0$, this implies that

$$\lim_{r \nearrow r_1 + \varepsilon} v_{\varepsilon r}(r, t) \text{ exists for } t > t^*$$

and thus

$$v_{\varepsilon r}(\cdot, t) \in C((r_0, r_1 + \varepsilon]), \qquad t > t^*.$$

Finally, we claim that for some $T \geqslant t^*$

$$v_{\varepsilon r}(r_1 + \varepsilon, T) < 0, \tag{A.6}$$

from which Lemma A follows at once.

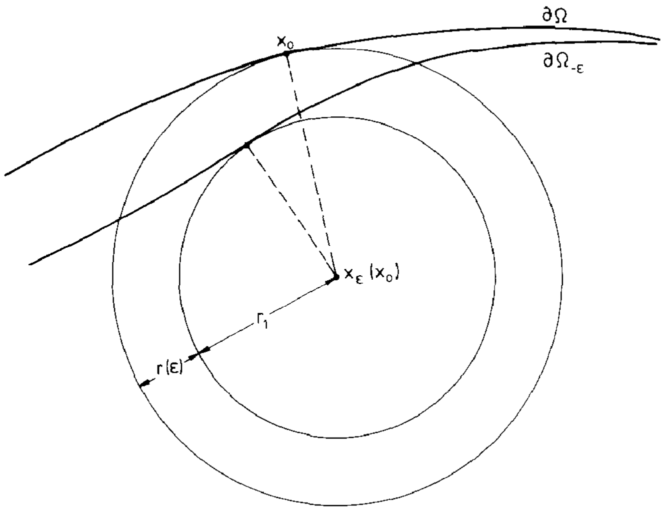
Suppose that (A.6) is not true, i.e.,

$$v_\varepsilon(r_1 + \varepsilon, t) = v_{\varepsilon r}(r_1 + \varepsilon, t) = 0 \qquad \text{for} \quad t \geqslant t^*. \tag{A.7}$$

Again we extend v_ε to $(r_1 + \varepsilon, r_1 + 1] \times [0, \infty)$ by 0. Let $\eta \in C^2([r_0, r_1 + 1] \times [t^*, \infty))$ be a test function which vanishes on $\{r_0\} \cup \{r_1 + 1\} \times [t^*, \infty)$. Multiplying the differential equation of Problem P_1 by η and integrating by parts over $(r_0, r_1 + 1) \times (t^*, t^* + \tau)$ $(\tau > 0)$ yield, from (A.7), that v_ε is a solution of Problem P_1 on $(r_0, r_1 + 1) \times [t^*, \infty)$. By (A.5), it also is a solution of Problem P_1 for $t \in [0, t^*)$. Hence $v_\varepsilon \equiv v_1$, which contradicts (A.3). Thus (A.6) holds and the proof of Lemma A is complete.

Proof of (A.1). Since $\partial\Omega \in C^3$, it satisfies the interior sphere condition. Combining this with the construction of the domains $\Omega_{-\varepsilon} \subset \Omega$, which we gave in the proof of Proposition 4.3, it follows that for some small $\varepsilon_2 > 0$ there exists a number $r_1 > 0$ such that: for any $0 < \varepsilon \leqslant \varepsilon_2$ and $x_0 \in \partial\Omega$, there exists a number $r(\varepsilon) > 0$ and a point $x_\varepsilon(x_0) \in \Omega_{-\varepsilon}$ such that

$$\overline{B(x_\varepsilon(x_0); r_1)} \cap \partial\Omega_{-\varepsilon} \text{ consists of one point}$$



and

$$\overline{B(x_\varepsilon(x_0); r_1 + r(\varepsilon))} \cap \partial\Omega = \{x_0\}.$$

In particular, since $\partial\Omega \in C^3$,

$$r(\varepsilon) \to 0 \qquad \text{as} \quad \varepsilon \searrow 0. \tag{A.8}$$

Let $\underline{\psi}_\varepsilon$ be as in Section 4. Since

$$\frac{\partial \bar{v}}{\partial \nu} \leqslant -\alpha_1 \qquad \text{on} \quad \partial\Omega,$$

for some $\alpha_1 > 0$, there exists a number $\varepsilon_3 \in (0, \varepsilon_2)$ such that for $0 < \varepsilon < \varepsilon_3$

$$\frac{\partial \underline{\psi}}{\partial \nu} \leqslant -\frac{1}{2}\alpha_1 \qquad \text{on} \quad \partial\Omega_{-\varepsilon}.$$

Hence there exists for some $r_0 \in (0, r_1)$ a function $p \in C([r_0, r_1 + 1]) \cap C^2(r_0, r_1)$ which satisfies (A_0), such that for any $0 < \varepsilon < \varepsilon_3$ and $x_0 \in \partial\Omega$

$$p(|x - x_\varepsilon(x_0)| \leqslant \underline{\psi}_\varepsilon(x) \qquad \text{for} \quad x \in B(x_\varepsilon(x_0); r_1 + r(\varepsilon)) \backslash B(x_\varepsilon(x_0); r_0).$$

Since in addition

$$v_\varepsilon(r_0, t) = p(r_0) \leqslant \underline{\psi}_\varepsilon(x) \leqslant v(x, t; \underline{\psi}_\varepsilon) \qquad \text{if} \quad |x - x_\varepsilon(x_0)| = r_0,$$

it follows from the comparison principle that for any $\varepsilon \in (0, \varepsilon_3)$ and $x_0 \in \partial\Omega$

$$v(x, t; \underline{\psi}_\varepsilon) \geqslant v_\varepsilon(|x - x_\varepsilon(x_0)|, t) \qquad \text{for}$$
$$x \in B(x_\varepsilon(x_0); r_1 + r(\varepsilon)) \backslash B(x_\varepsilon(x_0); r_0).$$

Thus, by (A.8) and Lemma A, (A.1) holds for some $\varepsilon_1 \in (0, \varepsilon_3)$.

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